Markowitz model: quadratic programing

László Losonczi

Debrecen University, Faculty of Economics and Business Administration

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Portfolio

Harry Markowitz received the 1990 Nobel Prize in Economics for his portfolio optimization model in which the tradeoff between risk and reward is explicitly treated.
(Reference: Robert J. Vanderbei, Linear Programing, Foundations and Extensions).

Given a collection of n potential investments R_i $(i = 1, \ldots, n)$ where R_i denotes the return (capital+interest) in the next time period on investment. In general, R_i is a random variable, although some investments may be essentially deterministic.

A **portfolio** is a collection of nonnegative numbers x_i ($i = 1, \ldots, n$) that sum to one. The return (on each dollar) one would obtain using a given portfolio is given by

$$R = \sum_{i=1}^{n} x_i R_i.$$

The reward associated with such a portfolio is defined as the expected value of the return:

$$\mathrm{E}\, R = \sum_{i=1}^{n} x_i \,\mathrm{E}\, R_i.$$

If reward alone were the issue, the problem would be trivial: simply put everything in the investment with the highest expected return. But unfortunately, investments with high reward typically also carry a high level of risk. That is, even though they are expected to do very well in the long run, they also tend to be erratic in the short term.

Markowitz defined the **risk** associated with an investment to be the **variance of the return**:

$$\operatorname{var} R = \mathrm{E}(R - \mathrm{E}\,R)^2 = \mathrm{E}\left(\sum_{i=1}^{n} x_i(R_i - \mathrm{E}\,R_i)\right)^2$$

$$= \mathrm{E}\left(\sum_{i=1}^{n} x_i \widetilde{R}_i\right)^2$$

where

$$\widetilde{R}_i = R_i - \mathrm{E}\,R_i.$$

One would like to maximize the reward while at the same time not incur excessive risk.

In the Markowitz model, one forms a linear combination of the mean and the variance (parametrized here by a positive μ) and get the following optimization problem:

$$\begin{aligned} &\text{maximize} && \sum_{i=1}^{n} x_i \,\mathrm{E}\, R_i - \mu\, \mathrm{E}\left(\sum_{i=1}^{n} x_i \widetilde{R}_i\right)^2 \\ &\text{subject to} && \sum_{i=1}^{n} x_i = 1 \\ & && x_i \geq 0 \qquad (i = 1, \ldots, n). \end{aligned} \tag{1}$$

μ is a positive parameter that represents the importance of risk relative to reward: high values of μ tend to minimize risk at the expense of reward, whereas low values put more weight on reward.
For example if $\mu = 0$ the we maximize the expected return and do not take into consideration the risks.

The maximum problem (1) is equivalent to the following minimization problem:

$$\begin{aligned} &\text{minimize} && -\sum_{i=1}^{n} x_i \operatorname{E} R_i + \mu \operatorname{E}\left(\sum_{i=1}^{n} x_i \widetilde{R}_i\right)^2 \\ &\text{subject to} && \sum_{i=1}^{n} x_i = 1 \\ & && x_i \geq 0 \qquad (i = 1, \dots, n). \end{aligned} \tag{2}$$

From the form of the objective function one can see that its quadratic part is positive semi-definite. Therefore (2) is a quadratic programing problem.

Expanding the square in the variance we get that

$$\mathrm{E}\left(\sum_{i=1}^{n} x_i \widetilde{R}_i\right)^2 = \mathrm{E}\left(\sum_{i=1}^{n} x_i \widetilde{R}_i\right)\left(\sum_{j=1}^{n} x_j \widetilde{R}_j\right) = \mathrm{E}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \widetilde{R}_i \widetilde{R}_j\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \,\mathrm{E}(\widetilde{R}_i \widetilde{R}_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j C_{ij}$$

where

$$C_{ij} = \mathrm{E}(\widetilde{R}_i \widetilde{R}_j)$$

is the covariance matrix. Hence, with the notation $r_i = \mathrm{E}\, R_i$ our problem has the form:

$$\begin{aligned} \text{minimize} \qquad & -\sum_{i=1}^{n} r_i x_i + \mu \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j C_{ij} \\ \text{subject to} \qquad & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0 \qquad (i = 1, \ldots, n) \end{aligned} \tag{3}$$

Solving this problem requires an estimate of the mean return for each of the investments as well as an estimate of the covariance matrix. However, these quantities are not known theoretically but instead **must be estimated by looking at historical data.**

The next table shows **annual returns from 1973 to 1994 for eight different possible investments for** $1\$$:

1. U.S. Three-Month T-Bills,
2. Government Long Bonds,
3. S& P 500,
4. Wilshire 500 (a collection of small company stocks),
5. NASDAQ Composite,
6. Lehman Brothers Corporate Bonds Index,
7. EAFE (a securities index for Europe, Asia, and the Far East),
8. Gold

For example 1$ invested on January 1, 1973 into U.S. Three-Month T-Bills has increased by December 31, 1973 to $1,075\$$.

Year	US 3-Month T-Bills	US Gov. Long Bonds	S&P 500	Wilshire 5000	NASDAQ Composite	Lehman Bros. Corp. Bonds	EAFE	Gold
1973	1.075	0.942	0.852	0.815	0.698	1.023	0.851	1.677
1974	1.084	1.020	0.735	0.716	0.662	1.002	0.768	1.722
1975	1.061	1.056	1.371	1.385	1.318	1.123	1.354	0.760
1976	1.052	1.175	1.236	1.266	1.280	1.156	1.025	0.960
1977	1.055	1.002	0.926	0.974	1.093	1.030	1.181	1.200
1978	1.077	0.982	1.064	1.093	1.146	1.012	1.326	1.295
1979	1.109	0.978	1.184	1.256	1.307	1.023	1.048	2.212
1980	1.127	0.947	1.323	1.337	1.367	1.031	1.226	1.296
1981	1.156	1.003	0.949	0.963	0.990	1.073	0.977	0.688
1982	1.117	1.465	1.215	1.187	1.213	1.311	0.981	1.084
1983	1.092	0.985	1.224	1.235	1.217	1.080	1.237	0.872
1984	1.103	1.159	1.061	1.030	0.903	1.150	1.074	0.825
1985	1.080	1.366	1.316	1.326	1.333	1.213	1.562	1.006
1986	1.063	1.309	1.186	1.161	1.086	1.156	1.694	1.216
1987	1.061	0.925	1.052	1.023	0.959	1.023	1.246	1.244
1988	1.071	1.086	1.165	1.179	1.165	1.076	1.283	0.861
1989	1.087	1.212	1.316	1.292	1.204	1.142	1.105	0.977
1990	1.080	1.054	0.968	0.938	0.830	1.083	0.766	0.922
1991	1.057	1.193	1.304	1.342	1.594	1.161	1.121	0.958
1992	1.036	1.079	1.076	1.090	1.174	1.076	0.878	0.926
1993	1.031	1.217	1.100	1.113	1.162	1.110	1.326	1.146
1994	1.045	0.889	1.012	0.999	0.968	0.965	1.078	0.990

Let $R_i(t)$ $(i = 1, \dots, n)$ denote the return on ith investment in year $1972 + t$. One way to estimate the mean $\mathrm{E}\, R_i$ is simply to take the average (arithmetic mean) of the historical returns:

$$r_i = \mathrm{E}\, R_i = \frac{1}{T} \sum_{t=1}^{T} R_i(t).$$

There are two drawbacks to this simple formula.

First, whatever happened in 1973 certainly has less bearing on the future than what happened in 1994. Hence, giving all the past returns equal weight puts too much emphasis on the distant past at the expense of the recent past. A better estimate is obtained by using a discounted sum:

$$\mathrm{E}\, R_i = \frac{\sum_{t=1}^{T} p^{T-t} R_i(t)}{\sum_{t=1}^{T} p^{T-t}}.$$

Here, p is a **discount factor**.

Putting $p = 0.9$ gives a weighted average that puts more weight on the most recent years. To see the effect of discounting the past, consider the Gold investment. The unweighted average return is 1.129, whereas the weighted average is 1.053. Most experts in 1995 felt that a 5.3% return represented a more realistic expectation than a 12.9% return. In the results that follow, **all expectations are estimated by computing weighted averages using** $p = 0.9$.

The **second issue** concerns the estimation of means (not variances). An investment that returns 1.1 one year and 0.9 the next has an (unweighted) average return of 1, that is, no gain or loss. However, one dollar invested will actually be worth $(1.1)(0.9) = 0.99$ at the end of the second year. While a 1% error is fairly small, consider what happens if the return is 2.0 one year and then 0.5 the next. Clearly, the value of one dollar at the end of the two years is $(2.0)(0.5) = 1$, but the average of the two returns is $(2.0 + 0.5)/2 = 1.25$. There is a very significant difference between an investment that is flat and one that yields a 25% return in two years. This is obviously an effect for which a correction is required.

The trick is to average the logarithm of the returns (and then exponentiate the average). Hence in the estimate we use the **weighted geometric mean**:

$$\mathrm{E}\, R_i = \exp\left(\frac{\sum_{t=1}^{T} p^{T-t} \ln R_i(t)}{\sum_{t=1}^{T} p^{T-t}}\right) = \left(\prod_{t=1}^{T} R_i(t)^{(p^{T-t})}\right)^{\frac{1}{\sum_{t=1}^{T} p^{T-t}}}$$

This estimate for Gold gives an estimate of its return at 2.9%, which is much more in line with the beliefs of experts (at least in 1995).

The elements of the covariance matrix C_{ij} can also be estimated based on historical data, for example by their arithmetic mean:

$$C_{ij} = \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} (R_i(t) - r_i)(R_j(s) - r_j) \qquad (i, j = 1, \ldots, n).$$

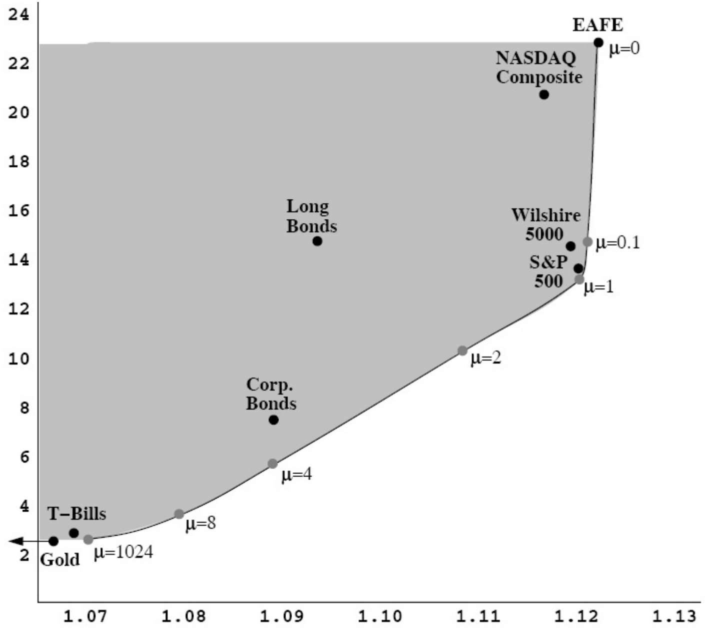
If we know the values of r_i, C_{ij} the with a given μ the quadratic programing problem (3) can be solved.

The next table gives the **optimal portfolios** at several values of μ **for the year 1995**.

μ	Gold	US 3-Month T-Bills	Lehman Bros. Corp. Bonds	NASDAQ Composite	S&P 500	EAFE	Mean	Std. Dev.
0.0						1.000	1.122	0.227
0.1					0.603	0.397	1.121	0.147
1.0					0.876	0.124	1.120	0.133
2.0		0.036	0.322		0.549	0.092	1.108	0.102
4.0		0.487	0.189		0.261	0.062	1.089	0.057
8.0		0.713	0.123		0.117	0.047	1.079	0.037
1024.0	0.008	0.933	0.022	0.016		0.022	1.070	0.028

At $\mu = 0$ our portfolio contains one single investment, EAFE, since (based on historical data) this has the highest expected value. For large μ, e.g. for $\mu = 1024$, $93,3\%$ of our portfolio has Government Long Bonds, and only $2,2\%$ EAFE, since the latter has large variance and Government Long Bonds has small variance (based on historical data).

Letting μ vary continuously we plot variance and expected value of the optimal portfolio on a curve :


24
22
20
18
16
14
12
10
8
6
4
2
EAFE
μ=0
NASDAQ
Composite
Long
Bonds
Wilshire
5000
μ=0.1
S&P
500
μ=1
μ=2
Corp.
Bonds
μ=4
T−Bills
μ=8
Gold
μ=1024
1.07
1.08
1.09
1.10
1.11
1.12
1.13

Here the expected values of the optimal portfolio are on the horizontal axis, and on the vertical axis the variances of the optimal portfolio (multiplied by 100) are given. The values of μ are plotted are the corresponding points, and also the points represented the 8 investments are given.

The curve above is called the **efficient frontier**. Any portfolio that produces a meanvariance combination that does not lie on the efficient frontier can be improved either by increasing its mean without changing the variance or by decreasing the variance without changing the mean. Hence, **one should only invest in portfolios that lie on the efficient frontier.**