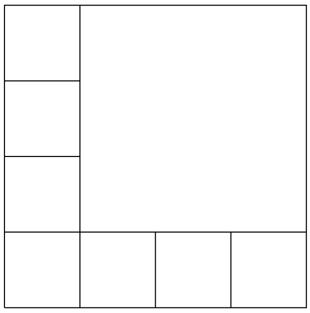

Exercise 3.1. Prove that $9^n - 1$ is divisible by 8 for all $n \in \mathbb{N}$.

Exercise 3.2. Prove that $5^{2n-1} + 1$ is divisible by 6 for all $n \in \mathbb{N}$.

Exercise 3.3. Prove the following identity by induction

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Exercise 3.4. Prove the following identity by induction

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Exercise 3.5. Prove the following identity by induction

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Exercise 3.6. Prove the following identity by induction

$$\sum_{i=1}^{n-1} i(i+1) = \frac{(n-1)\,n\,(n+1)}{3}.$$

Exercise 3.7. Prove the following identity by induction

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

Exercise 3.8. Let $\{\, a_n \,\}$ be a sequence defined by

$$\begin{aligned} a_1 &= 1, \\ a_2 &= 8, \\ a_n &= a_{n-1} + 2a_{n-2}, \quad n \geq 3. \end{aligned}$$

Prove that

$$a_n = \frac{3}{2} \cdot 2^n + 2 \cdot (-1)^n.$$

Exercise 3.9. Prove by induction that the number

$$\left(\frac{3-\sqrt{33}}{2}\right)^n + \left(\frac{3+\sqrt{33}}{2}\right)^n$$

is an integer which is divisible by 3 for all $n \in \mathbb{N}$.

Exercise 3.10. Let $\{ a_n \}$ be a sequence defined by

$$\begin{aligned} a_1 &= \sqrt{2}, \\ a_n &= \sqrt{2 + a_{n-1}} \quad n \geq 2. \end{aligned}$$

Prove by induction that $a_n \leq 2$ for all $n \geq 1$.

Exercise 3.11. Prove that for all $n \in \mathbb{N}$ there exists an n-digit integer

$$a_1 a_2 \ldots a_n$$

whose digits are either 1 or 2 and it is divisible by 2^n.

Exercise 3.12. Let F_n be a sequence defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}, n \geq 3$ (this sequence is the so-called Fibonacci sequence). Prove by induction the following identities.

(a) $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$,

(b) $F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$,

(c) $F_1 + F_3 + \ldots + F_{2n-1} = F_{2n}$,

(d) $F_2 + F_4 + \ldots + F_{2n} = F_{2n+1} - 1$.

Exercise 3.13. Prove the following properties of Fibonacci numbers.

(a) Prove that F_{3n} is even for all $n \in \mathbb{N}$.

(b) Prove that F_{5n} is divisible by 5 for all $n \in \mathbb{N}$.